NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

ASYMPTOTIC SOLUTION OF A NONLINEAR DIFFERENTIAL EQUATION OF SECOND ORDER

By G. E. Kuzmak

Translation

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ABSTRACT

The solution of the nonlinear differential equation

$$\frac{d^2y}{dt^2} + a(\tau)y - b(\tau)y^3 = 0 \qquad (\tau = \varepsilon t)$$

is considered. The object of the study is to determine the first terms of the asymptotic expansion of the solution for small ε. The method used is to perturb the solution obtained from a certain similar equation, in this case the equation for the elliptic sine, which differs from the above equation only in that the slowly varying coefficients a and b are replaced by constant values.

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

ASYMPTOTIC SOLUTION OF A NONLINEAR DIFFERENTIAL EQUATION OF SECOND ORDER*

By G. E. Kuzmak

In this paper there is considered the equation

$$\frac{d^2y}{dt^2} + a(\tau)y - b(\tau)y^3 = 0 \qquad (\tau = \varepsilon t) \tag{0.1}$$

frequently encountered in various technical problems. The object of the study is to determine the first term of the asymptotic expansion of the solution for small ε. An analogous problem was considered in references 1 and 2.

The method here applied consists in expressing the solution of this equation in terms of the solution of a certain analogous equation ("standard equation"). This method of "standard equations" has been worked out in the case of linear equations (refs. 3 and 4); however, it apparently has not been applied to nonlinear equations.

In the case under consideration, there has been chosen as "standard" the equation for the elliptic sine (see eq. (1.7)), which differs from (0.1) only in that the slowly varying coefficient appearing in it is replaced by a constant magnitude.

1. Computation of the first terms of the asymptotic expansion. - In order to find the asymptotic expansion of the solution of equation (0.1), it is first necessary to represent the dependence of the solution on the time and a small parameter.

In this case, if the coefficients a and b are constant, the solution of equation (0.1) is the function $A\ \mathrm{sn}(\varphi t,\nu)$, where A, φ, and ν are certain constants.

In the case of slowly varying coefficients, the solution will be some function close to the elliptic sine function, but with a variable

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amplitude, frequency, and modulus. Hence, in this case the solution can be represented in the form[1]

$$y = A(\tau)\operatorname{sn}[T(\tau)\omega,\ \nu(\tau)] + O(\varepsilon) \qquad (\omega = \int\varphi(\tau)dt) \tag{1.1}$$

Examination of (1.1) shows that the solution depends on the variables τ and ω, the derivatives of which $d\tau/dt = \varepsilon$ and $d\omega/dt = \varphi(\tau)$ are magnitudes of different order with respect to ε.

Setting $y = y(\tau, \omega, \varepsilon)$, we shall seek y as a function of the two variables τ and ω. Equation (0.1) assumes the form

$$\varphi^2\frac{\partial^2 y}{\partial\omega^2} + \varepsilon\left(2\varphi\frac{\partial^2 y}{\partial\omega\partial\tau} + \varphi'\frac{\partial y}{\partial\omega}\right) + \varepsilon^2\frac{\partial^2 y}{\partial\tau^2} + a(\tau)y - b(\tau)y^3 = 0 \tag{1.2}$$

We shall seek the solution with an accuracy up to terms of order ε^2:

$$y = y_0(\tau, \omega) + \varepsilon y_1(\tau, \omega)$$

Substituting this expression in equation (1.2), we obtain

$$\left\{\varphi^2\frac{\partial^2 y_0}{\partial\omega^2} + a(\tau)y_0 - b(\tau)y_0^3\right\} + \varepsilon\left\{\varphi^2\frac{\partial^2 y_1}{\partial\omega^2} + \left[a(\tau) - 3b(\tau)y_0^2\right]y_1 + 2\varphi\frac{\partial^2 y_0}{\partial\omega\partial\tau} + \varphi'\frac{\partial y_0}{\partial\omega}\right\} + \varepsilon^2\Delta(t) = 0 \tag{1.3}$$

where

$$\Delta(t) = -3b(\tau)y_0y_1^2 + 2\varphi\frac{\partial^2 y_1}{\partial\omega\partial\tau} + \varphi'\frac{\partial y_1}{\partial\omega} + \frac{\partial^2 y_0}{\partial\tau^2} + \varepsilon\left[\frac{\partial^2 y_1}{\partial\tau^2} - b(\tau)y_1^3\right]$$

In order to satisfy this equation with an accuracy up to ε^2, it is sufficient to require that the functions $y_0(\tau,\omega)$ and $y_1(\tau,\omega)$ satisfy the equations

$$\begin{gathered}\varphi^2\frac{\partial^2 y_0}{\partial\omega^2} + a(\tau)y_0 - b(\tau)y_0^3 = 0 \\ \varphi^2\frac{\partial^2 y_1}{\partial\omega^2} + [a(\tau) - 3b(\tau)y_0^2]y_1 = -2\dot{\varphi}\frac{\partial^2 y_0}{\partial\omega\partial\tau} - \varphi'\frac{\partial y_0}{\partial\omega}\end{gathered} \tag{1.4}$$

[1]Throughout the present paper we shall denote by ν the square of the modulus of the elliptic functions and integrals. In this connection, in place of the usual notations $\operatorname{sn}(u,\sqrt{\nu})$, $\operatorname{cn}(u,\sqrt{\nu})$, $\operatorname{dn}(u,\sqrt{\nu})$, $K(\sqrt{\nu})$, and $E(\sqrt{\nu})$, we shall write $\operatorname{sn}(u,\ \nu)$, $\operatorname{cn}(u,\nu)$, $\operatorname{dn}(u,\nu)$, $K(\nu)$, and $E(\nu)$.

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To find the solution of the first equation of this system, we put

$$y_0 = A(\tau)\mathrm{sn}[T(\tau)\omega,\nu(\tau)] = A(\tau)\mathrm{sn}\ u, \qquad u = T(\tau)\omega \tag{1.5}$$

From (1.5) we have

$$\frac{\partial y_0}{\partial\omega} = AT\frac{\partial\ \mathrm{sn}\ u}{\partial u}, \qquad \frac{\partial^2 y_0}{\partial\omega^2} = AT^2\frac{\partial^2\mathrm{sn}\ u}{\partial u^2} \tag{1.6}$$

Substituting these formulas in the equation for $y_0(\tau,\omega)$, we obtain

$$AT^2\varphi^2\frac{\partial^2\mathrm{sn}\ u}{\partial u^2} + a(\tau)A\ \mathrm{sn}\ u - b(\tau)A^3\mathrm{sn}^3u = 0$$

Replacing in this equation $\partial^2\mathrm{sn}\ u/\partial u^2$ with the aid of the following equation (ref. 5)

$$\frac{\partial^2\mathrm{sn}\ u}{\partial u^2} + (1+\nu)\mathrm{sn}\ u - 2\nu\ \mathrm{sn}^3u = 0 \tag{1.7}$$

we have

$$[-T^2\varphi^2(1+\nu) + a(\tau)] + \mathrm{sn}^2u[T^2\varphi^2 2\nu - b(\tau)A^2] = 0$$

In order to satisfy this relation, let us put

$$\varphi^2T^2(1+\nu) = a(\tau) \qquad \varphi^2T^2 2\nu = b(\tau)A^2 \tag{1.8}$$

The additional two relations for determining $\varphi(\tau)$, $T(\tau)$, $A(\tau)$, and $\nu(\tau)$ we obtain from the condition of the periodicity of the function $y_1(\tau,\omega)$ with respect to ω.

As will be shown in section 2, this condition is sufficient for the terms that we neglected in (1.3) to be $O(\varepsilon^2)$ on the time interval $0 \le t \le T_0/\varepsilon$.

We proceed further to the determination of the function $y_1(\tau,\omega)$.

From (1.6) we have

$$\frac{\partial^2 y_0}{\partial\omega\partial\tau} = (AT)'\frac{\partial\ \mathrm{sn}\ u}{\partial u} + AT\left(\frac{\partial^2\mathrm{sn}\ u}{\partial u^2}\frac{\partial u}{\partial\tau} + \frac{\partial^2\mathrm{sn}\ u}{\partial u\partial\nu}\nu'\right) \tag{1.9}$$

where the prime denotes differentiation with respect to τ. We note that

$$\frac{\partial u}{\partial \tau} = u\,\frac{T'}{T}, \qquad \frac{\partial u}{\partial \omega} = T \tag{1.10}$$

Using relations (1.9) and (1.10), we rewrite the equation for $y_1(\tau,\omega)$ in the form

$$\varphi^2 T^2\,\frac{\partial^2 y_1}{\partial u^2} + [a(\tau) - 3b(\tau)y_0{}^2]y_1 = F(\tau,u) \tag{1.11}$$

where

$$F(\tau,u) = -\,[2\varphi(AT)' + \varphi' AT]\,\frac{\partial^2 \mathrm{sn}\ u}{\partial u} - 2\varphi AT\left[\frac{\partial\ \mathrm{sn}\ u}{\partial u^2}\,u(\ln T)' + \frac{\partial\ \mathrm{sn}\ u}{\partial u\partial \nu}\,\nu'\right] \tag{1.12}$$

Let us find the solution of the homogeneous equation. Differentiating for this purpose the first equation of the system (1.4),

$$\varphi^2 T^2\,\frac{\partial^2}{\partial u^2}\left(\frac{\partial y_0}{\partial u}\right) + [a(\tau) - 3b(\tau)y_0{}^2]\,\frac{\partial y_0}{\partial u} = 0 \tag{1.13}$$

whence it is seen that the function

$$\upsilon = \frac{\partial\ \mathrm{sn}\ u}{\partial u}$$

is the required function.

In order to write out the expression for the function $y_1(\tau, u)$, we put

$$y_1(\tau, u) = \upsilon\int_0^u E(\tau, u)du \tag{1.14}$$

Substituting this expression in equation (1.11), after several transformations we obtain

$$E(\tau,u) = \frac{1}{\upsilon^2}\int_0^u \frac{F(\tau, u)}{\varphi^2 T^2}\,\upsilon du$$

Replacing $F(\tau, u)$ with the aid of (1.12), we obtain

$$E(\tau, u) = -\,\frac{1}{(\partial\ \mathrm{sn}\ u/\partial u)^2}\left\{\frac{2\varphi(AT)' + \varphi' AT}{\varphi^2 T^2}\,J_1 + \frac{2A(\ln T)'}{\varphi T}\,J_2 + \frac{2A\nu'}{\varphi T}\,J_3\right\} \tag{1.15}$$

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where

$$J_1 = \int_0^u \left(\frac{\partial \text{ sn } u}{\partial u}\right)^2 du, \quad J_2 = \int_0^u u \frac{\partial^2 \text{sn } u}{\partial u^2} \frac{\partial \text{ sn } u}{\partial u} du, \quad J_3 = \int_0^u \frac{\partial \text{ sn } u}{\partial u \partial \nu} \frac{\partial \text{ sn } u}{\partial u} du \tag{1.16}$$

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We further compute the derivative of the elliptic sine with respect to the modulus. For this purpose we make use of the known relations (ref. 5) for

$$\left(\frac{\partial \text{ sn } u}{\partial u}\right)^2 = 1 - (1 + \nu)\text{sn}^2 u + \nu \,\text{sn}^4 u \tag{1.17}$$

Differentiating it with respect to ν,

$$\frac{\partial \text{ sn } u}{\partial u} \frac{\partial^2 \text{sn } u}{\partial u \partial \nu} = [\, -(1 + \nu)\text{sn } u + 2\nu \text{sn}^3 u \,] \frac{\partial \text{ sn } u}{\partial \nu} + \frac{-\text{sn}^2 u + \text{sn}^4 u}{2}$$

$$= \frac{\partial^2 \text{sn } u}{\partial u^2} \frac{\partial \text{ sn } u}{\partial \nu} - \frac{\text{sn}^2 u \text{ cn}^4 u}{2}$$

The obtained relation is a linear differential equation of the first order with respect to the function $\partial \text{sn } u/\partial \nu$. Solving it, we have

$$\frac{\partial \text{ sn } u}{\partial \nu} = - \frac{1}{2} \frac{\partial \text{ sn } u}{\partial u} \int_0^u \frac{\text{sn}^2 u}{\text{dn}^2 u} du \tag{1.18}$$

In order to write out the conditions of periodicity, we transform the integrals J_2 and J_3 in such manner as to represent them in the form of a sum of integrals of positive functions

$$J_2 = \frac{1}{2}\int_0^u u \frac{\partial}{\partial u}\left(\frac{\partial \text{ sn } u}{\partial u}\right)^2 du = \frac{1}{2}\left[u\left(\frac{\partial \text{ sn } u}{\partial u}\right)^2 - J_1\right]$$

$$J_3 = \int_0^u \frac{\partial \text{ sn } u}{\partial u} \frac{\partial}{\partial u}\left(\frac{\partial \text{ sn } u}{\partial \nu}\right) du = \left(\frac{\partial \text{ sn } u}{\partial u} \frac{\partial \text{ sn } u}{\partial \nu}\right)\Bigg|_0^u - \int_0^u \frac{\partial \text{ sn } u}{\partial \nu} \frac{\partial \text{ sn } u}{\partial u^2} du \tag{1.19}$$

Making use of (1.18) for $\partial \text{sn } u/\partial \nu$ and integrating by parts, we reduce the expression for J_3 to the form

$$J_3 = -\frac{1}{4}\left(\frac{\partial \text{ sn } u}{\partial u}\right)^2 \int_0^u \frac{\text{sn}^2 u}{\text{dn}^2 u}\, du - \frac{1}{4}\int_0^u \left(\frac{\partial \text{ sn } u}{\partial u}\right)^2 \frac{\text{sn}^2 u}{\text{dn}^2 u}\, du \tag{1.20}$$

In these formulas (see ref. 5)

$$\frac{\partial \text{ sn } u}{\partial u} = \text{cn } u \text{ dn } u \tag{1.21}$$

Substituting formulas (1.19) and (1.20) in the expression for $E(\tau, u)$, we have

$$E(\tau, u) = -\frac{A\nu'}{\varphi T}\left\{\left[\frac{d \ln T}{d\nu} u - \frac{1}{2}\int_0^u \frac{\text{sn}^2 u}{\text{dn}^2 u}\, du\right] + \frac{1}{\text{cn}^2 u \text{ dn}^2 u}\left[\frac{d \ln A^2 \varphi T}{d\nu}\int_0^u \text{cn}^2 u \text{ dn}^2 u du - \frac{1}{2}\int_0^u \text{sn}^2 u \text{ cn}^2 u du\right]\right\} \tag{1.22}$$

From this it follows that if the functions $d \ln T/d\nu$ and $d \ln A^2\varphi T/d\nu$ satisfy the conditions

$$\frac{d \ln T}{d\nu} 2K(\nu) = \frac{1}{2}\int_u^{u+2K(\nu)} \frac{\text{sn}^2 u}{\text{dn}^2 u}\, du, \qquad \frac{d \ln A^2 T}{d\nu}\int_u^{u+2K(\nu)} \text{cn}^2 u \text{ dn}^2 u du = \frac{1}{2}\int_u^{u+K(\nu)} \text{sn}^2 u \text{ cn}^2 u du \tag{1.23}$$

then the following relation holds:

$$E(\tau, u) = E(\tau, u + 2K(\nu)) \tag{1.24}$$

where $K(\nu)$ is the complete elliptic integral of the first kind.

In virtue of the periodicity of the functions under the integral signs in (1.23) and their symmetry with respect to the point $u = K(\nu)$, we have

$$\frac{d\ \ln T}{d\nu} = \frac{1}{2K(\nu)}\int_0^{K(\nu)} \frac{\mathrm{sn}^2 u}{\mathrm{dn}^2 u}\, du$$

$$\frac{d\ \ln A^2\varphi T}{d\nu} = \frac{1}{2}\int_0^{K(\nu)} \mathrm{sn}^2 u\ \mathrm{cn}^2 u\,du \Big/ \int_0^{K(\nu)} \mathrm{dn}^2 u\ \mathrm{cn}^2 u\,du \tag{1.25}$$

Introducing the new variable of integration $\zeta = \mathrm{sn}\ u$ and using the following formulas (ref. 5)

$$\mathrm{cn}^2 u = 1 - \zeta^2, \qquad \mathrm{dn}^2 u = 1 - \nu\zeta^2, \qquad \frac{\partial u}{\partial \zeta} = \frac{1}{\sqrt{(1 - \zeta^2)(1 - \nu\zeta^2)}}$$

it is not difficult to transform these relations to the form

$$\frac{d\ \ln T}{d\nu} = \frac{1}{K(\nu)}\frac{dK(\nu)}{d\nu}, \qquad \frac{d\ \ln A^2\varphi T}{d\nu} = -\frac{1}{L(\nu)}\frac{dL(\nu)}{d\nu} \tag{1.26}$$

where

$$K(\nu) = \int_0^1 \frac{d\zeta}{\sqrt{(1 - \zeta^2)(1 - \nu\zeta^2)}}, \qquad L(\nu) = \int_0^1 \sqrt{(1 - \zeta^2)(1 - \nu\zeta^2)}\, d\zeta \tag{1.27}$$

Whence, evidently,

$$T(\tau) = K(\nu(\tau)), \qquad A^2(\tau)\varphi(\tau)T(\tau)L(\nu(\tau)) = \mathrm{const} \tag{1.28}$$

From (1.22) we have $E(\tau, u) = -E(\tau, -u)$, whence

$$\int_{-K(\nu)}^{K(\nu)} E(\tau, u)du = 0$$

From the periodicity of $E(\tau, u)$ (see (1.24)), it follows that

$$\int_u^{u+2K(\nu)} E(\tau, u)du = \int_{-K(\nu)}^{K(\nu)} E(\tau, u)du$$

Consideration of these relations and formula (1.14) permits the conclusion that $y_1(\tau, u) = y_1(\tau, u + 4K(\nu))$.

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For what follows we introduce the following notations:

$$\rho(\tau) = \frac{A(\tau)\nu'(\tau)}{K(\nu(\tau))\varphi(\tau)_1}, \qquad S(\nu, \omega) = \operatorname{sn}\,[K(\nu)\omega, \nu] \tag{1.29}$$

$$G(\nu,\omega) = -K(\nu)\operatorname{cn} K(\nu)\omega \operatorname{dn} K(\nu)\omega \left\{ \int_0^{\omega} \left[\frac{dK(\nu)}{d\nu}\,\omega_1 - \frac{K(\nu)}{2}\int_0^{\omega_1} \frac{\operatorname{sn}^2 K(\nu)\omega_2}{\operatorname{dn}^2 K(\nu)\omega_2}\, d\omega_2 \right] d\omega_1 + \right.$$

$$\int_0^{\omega} \frac{K(\nu)}{\operatorname{cn}^2 K(\nu)\omega_1 \operatorname{dn}^2 K(\nu)\omega_1} \left[-\frac{d \ln L(\nu)}{d\nu} \int_0^{\omega_1} \operatorname{cn}^2 K(\nu)\omega_2 \operatorname{dn}^2 K(\nu)\omega_2\, d\omega_2 \right.$$

$$\left. \left. - \frac{1}{2}\int_0^{\omega_1} \operatorname{sn}^2 K(\nu)\omega_2 \operatorname{cn}^2 K(\nu)\omega_2\, d\omega_2 \right] d\omega_1 \right\} \tag{1.30}$$

In these notations, on account of (1.5), (1.14), and (1.22), the formulas for $y_0(\tau, \omega)$ and $y_1(\tau, \omega)$ assume the form

$$y_0(\tau, \omega) = A(\tau)\, S[\nu(\tau), \omega], \qquad y_1(\tau, \omega) = \rho(\tau)\, G[\nu(\tau), \omega] \tag{1.31}$$

In order to determine $y_0(\tau, \omega)$ and $y_1(\tau, \omega)$ as functions of t, it is necessary in formulas (1.31) to replace τ and ω with the aid of the relations

$$\tau = \varepsilon t, \qquad \omega = \int \varphi(\tau)\,dt$$

2. Theorems on estimates. - Lemma 1: Let

$$\frac{\partial^2 y}{\partial\omega^2} + m(\omega)y = n(\omega), \qquad y\Big|_{\omega=0} = \alpha, \qquad \frac{\partial y}{\partial\omega}\Big|_{\omega=0} = \beta$$

If the functions $m(\omega)$ and $n(\omega)$ are bounded for $\omega \geq 0$ so that $|m(\omega)| \leq M$ and $|n(\omega)| \leq N$, then for $y(\omega)$ and $\partial y/\partial\omega$ for $\omega \geq 0$, the following estimates hold:

$$\left.\begin{aligned} |y(\omega)| &\leq |\alpha| \operatorname{ch} \sqrt{M}\,\omega + \frac{|\beta|}{\sqrt{M}} \operatorname{sh} \sqrt{M}\,\omega + \frac{N}{M}(\operatorname{ch}\sqrt{M}\,\omega - 1) \\ \left|\frac{\partial y}{\partial\omega}\right| &\leq |\alpha| \sqrt{M} \operatorname{sh}\sqrt{M}\,\omega + |\beta| \operatorname{ch}\sqrt{M}\,\omega + \frac{N}{\sqrt{M}} \operatorname{sh}\sqrt{M}\,\omega \end{aligned}\right\} \tag{2.1}$$

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The proof can easily be conducted with the aid of the method of successive approximations.

Lemma 2: Let

$$1 - Q^2 \leq \nu \leq 1 - q^2 \tag{2.2}$$

where q is an arbitrarily small number and Q an arbitrarily large number.

Then for $-\infty < \omega < \infty$, the following estimates hold:

$$\left.\begin{aligned}
&|S(\nu, \omega)| \leq 1, \quad \left|\frac{\partial S(\nu, \omega)}{\partial \nu}\right| \leq C_\nu(q, Q), \quad \left|\frac{\partial^2 S(\nu, \omega)}{\partial \nu^2}\right| \leq C_{\nu,\nu}(q, Q)\\
&|G(\nu, \omega)| \leq D(q, Q), \quad \left|\frac{\partial G(\nu, \omega)}{\partial \omega}\right| \leq D_\omega(q, Q)\\
&\left|\frac{\partial G(\nu, \omega)}{\partial \nu}\right| \leq D_\nu(q, Q), \quad \left|\frac{\partial^2 G(\nu, \omega)}{\partial \omega \partial \nu}\right| \leq D_{\omega,\nu}(q, Q)\\
&\qquad\qquad \left|\frac{\partial^2 G(\nu, \omega)}{\partial \nu^2}\right| \leq D_{\nu,\nu}(q, Q)
\end{aligned}\right\} \tag{2.3}$$

To prove this we note first of all that for condition (2.2) it follows from the definition (1.27) of the functions $K(\nu)$ and $L(\nu)$ that

$$\left|\frac{d^r K(\nu)}{d\nu^r}\right| \leq E_r(q, Q), \quad \left|\frac{d^r L(\nu)}{d\nu^r}\right| \leq F_r(q, Q) \qquad (r=0,1,2,3,\ldots) \tag{2.4}$$

In virtue of the fact that, for the values ν and ω considered, the functions $\operatorname{sn} K(\nu)\omega$, $\operatorname{cn} K(\nu)\omega$, $\operatorname{dn} K(\nu)\omega$ assume only real values, there follows from the known formulas

$$\operatorname{sn}^2 K(\nu)\omega + \operatorname{cn}^2 K(\nu)\omega = 1, \qquad \operatorname{dn}^2 K(\nu)\omega + \nu \operatorname{sn}^2 K(\nu)\omega = 1$$

the estimates

$$|\operatorname{sn} K(\nu)\omega| \leq 1, \qquad |\operatorname{cn} K(\nu)\omega| \leq 1, \qquad |\operatorname{dn} K(\nu)\omega| \leq \sqrt{1 + |\nu|}$$

Using these inequalities we obtain (see (1.21) and (1.29))

$$|S(\nu, \omega)| \leq 1, \qquad \left|\frac{\partial S(\nu, \omega)}{\partial \omega}\right| \leq K(\nu)\sqrt{1 + |\nu|} \tag{2.5}$$

We shall now prove inequalities (2.3) for values of ω contained in the interval $0 \leq \omega \leq 4$. From (1.7) we have

$$\frac{\partial^2 S}{\partial\omega^2} + K^2(\nu)[(1+\nu)\,S - 2\nu S^3] = 0, \qquad S(\nu,\,\omega)\Big|_{\omega=0} = 0, \qquad \frac{\partial S(\nu,\,\omega)}{\partial\omega}\Big|_{\omega=0} = K(\nu)$$

Differentiating this equation and the initial conditions twice with respect to ν, we obtain

$$\left.\begin{aligned} &\frac{\partial^2}{\partial\omega^2}\left(\frac{\partial S}{\partial\nu}\right) + m(\nu,\,\omega)\left(\frac{\partial S}{\partial\nu}\right) = -\frac{dK^2(\nu)(1+\nu)}{d\nu}\,S + 2\,\frac{dK^2(\nu)\nu}{d\nu}\,S^3 \\ &\frac{\partial S(\nu,\,\omega)}{\partial\nu}\Big|_{\omega=0} = 0, \qquad \frac{\partial}{\partial\omega}\left(\frac{\partial S(\nu,\,\omega)}{\partial\nu}\right)\Big|_{\omega=0} = \frac{dK(\nu)}{d\nu} \end{aligned}\right\} \quad (2.6)$$

$$\left.\begin{aligned} &\frac{\partial^2}{\partial\omega^2}\left(\frac{\partial^2 S}{\partial\nu^2}\right) + m(\nu,\,\omega)\left(\frac{\partial^2 S}{\partial\nu^2}\right) = \Big[-\frac{dK^2(\nu)(1+\nu)}{d\nu} + 6S^2\,\frac{dK^2(\nu)\nu}{d\nu} \\ &\qquad - \frac{\partial m(\nu,\,\omega)}{\partial\nu}\Big]\frac{\partial S}{\partial\nu} - \frac{d^2K^2(\nu)(1+\nu)}{d\nu^2}\,S + 2\,\frac{d^2K^2(\nu)\nu}{d\nu^2}\,S^3 \\ &\frac{\partial^2 S(\nu,\,\omega)}{\partial\nu^2}\Big|_{\omega=0} = 0, \qquad \frac{\partial}{\partial\omega}\left(\frac{\partial^2 S(\nu,\,\omega)}{\partial\nu^2}\right)\Big|_{\omega=0} = \frac{d^2K(\nu)}{d\nu^2} \end{aligned}\right\} \quad (2.7)$$

where

$$m(\nu,\,\omega) = K^2(\nu)[(1+\nu) - 6\nu S^2(\nu,\,\omega)]$$

In virtue of inequalities (2.4) and (2.5), in equation (2.6) the coefficients, the right side, and the initial conditions are bounded. Hence, with the aid of lemma 1 it is possible to specify constants $C_\nu(q,Q)$ and $c_{\nu,\omega}(q,Q)$ such that for $0 \leq \omega \leq 4$

$$\left|\frac{\partial S(\nu,\,\omega)}{\partial\nu}\right| \leq C_\nu(q,\,Q), \qquad \left|\frac{\partial^2 S(\nu,\,\omega)}{\partial\omega\partial\nu}\right| \leq C_{\nu,\omega}(q,\,Q) \qquad (2.8)$$

Further, on account of the inequalities (2.4), (2.5), and (2.8), in equation (2.7) the coefficients, the right side, and the initial conditions are bounded. Hence, also with the aid of lemma 1, it is possible to specify constants $C_{\nu,\nu}(q,Q)$, and $C_{\nu,\nu,\omega}(q,Q)$ such that for $0 \leq \omega \leq 4$

$$\left|\frac{\partial^2 S(\nu, \omega)}{\partial \nu^2}\right| \leq C_{\nu,\nu}(q, Q), \qquad \left|\frac{\partial^3 S(\nu, \omega)}{\partial\omega\partial\nu^2}\right| \leq C_{\nu,\nu,\omega}(q, Q) \quad (2.9)$$

Further, from formulas (1.4) and (1.31), we have

$$\left.\begin{aligned} &\frac{\partial^2 G}{\partial\omega^2} + m(\nu, \omega)\, G = K(\nu)\left[\frac{d \ln L(\nu)K(\nu)}{d\nu}\,\frac{\partial S}{\partial\omega} - 2\,\frac{\partial^2 S}{\partial\omega\partial\nu}\right] \\ &G(\nu, \omega)\Big|_{\omega=0} = 0 \qquad \frac{\partial G(\nu, \omega)}{\partial\omega}\Big|_{\omega=0} = 0 \end{aligned}\right\} \quad (2.10)$$

Differentiating these relations twice with respect to ν, we obtain

$$\left.\begin{aligned} &\frac{\partial^2}{\partial\omega^2}\left(\frac{\partial G}{\partial\nu}\right) + m(\nu,\omega)\left(\frac{\partial G}{\partial\nu}\right) = -\frac{\partial m}{\partial\nu}\,G + \frac{\partial}{\partial\nu}\left\{K(\nu)\left[\frac{d \ln L(\nu)K(\nu)}{d\nu}\,\frac{\partial S}{\partial\omega} - 2\,\frac{\partial^2 S}{\partial\omega\partial\nu}\right]\right\} \\ &\frac{\partial G(\nu, \omega)}{\partial\nu}\Big|_{\omega=0} = 0, \qquad \frac{\partial}{\partial\omega}\left(\frac{\partial G(\nu, \omega)}{\partial\nu}\right)\Big|_{\omega=0} = 0 \end{aligned}\right\} \quad (2.11)$$

$$\left.\begin{aligned} &\frac{\partial^2}{\partial\omega^2}\left(\frac{\partial^2 G}{\partial\nu^2}\right) + m(\nu, \omega)\left(\frac{\partial^2 G}{\partial\nu^2}\right) \\ &= -\frac{\partial m}{\partial\nu}\,\frac{\partial G}{\partial\nu} - \frac{\partial}{\partial\nu}\left[\frac{\partial m}{\partial\nu}\,G\right] + \frac{\partial^2}{\partial\nu^2}\left\{K(\nu)\left[\frac{d \ln L(\nu)K(\nu)}{d\nu}\,\frac{\partial S}{\partial\omega} - 2\,\frac{\partial^2 S}{\partial\omega\partial\nu}\right]\right\} \\ &\frac{\partial^2 G(\nu, \omega)}{\partial\nu^2}\Big|_{\omega=0} = 0, \qquad \frac{\partial}{\partial\omega}\left(\frac{\partial^2 G(\nu, \omega)}{\partial\nu^2}\right)\Big|_{\omega=0} = 0 \end{aligned}\right\} \quad (2.12)$$

In virtue of inequalities (2.4), (2.5), (2.8), and (2.9) for the function $S(\nu,\omega)$ and its derivatives we can, as before, estimate the function $G(\nu,\omega)$ and its derivatives, applying successively lemma 1 and the equations (2.10), (2.11), and (2.12).

Thus, the inequalities (2.3) have been proved for a finite interval of variation of ω. In order to establish them for $-\infty < \omega < \infty$, we note that the functions $S(\nu,\omega)$ and $G(\nu,\omega)$ have a period with respect to ω not depending on ν and equal to 4 (see (1.29)). In virtue of this, all derivatives of these functions with respect to ν and ω are likewise periodic with respect to ω with the same period. (In order to convince oneself of this it is sufficient to differentiate the equations $S(\nu,\omega) = S(\nu,\omega + 4)$, $G(\nu,\omega) = G(\nu,\omega + 4)$ the required number of times with respect to ν and ω.) Hence the estimates established for $0 \leq \omega \leq 4$ are valid for $-\infty < \omega < \infty$. Thus the lemma has been proven.

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Let us further formulate the conditions that must be imposed on $a(\tau)$ and $b(\tau)$ in order that the terms that we neglect in equation (1.3) should be of the order ε^2 in the time interval $0 \leq t \leq T_0/\varepsilon$.

Theorem I: If the functions $a(\tau)$ and $b(\tau)$ are such that the functions $\nu(\tau)$, $A(\tau)$, and $\varphi(\tau)$ are determined from the system of equations (1.8) and (1.28) such that for $0 \leq t \leq T_0/\varepsilon$:

(1) The functions $\nu(\tau)$, $\varphi(\tau)$, $A(\tau)$, and $\rho(\tau)$ are bounded together with their first and second derivatives;

(2) $1 - Q^2 \leq \nu(\tau) < 1 - q^2$, then the function $y^*(t) = y_0(t) + \varepsilon y_1(t)$ for $0 \leq t \leq T_0/\varepsilon$ satisfies equation (0.1) with an accuracy up to terms $O(\varepsilon^2)$.

To prove this we must show that the function $\Delta(t)$ (see (1.3)) is bounded for $0 \leq t \leq T_0/\varepsilon$.

Let us write out the expressions

$$\left.\begin{aligned}
&y_0(t) = A(\tau)\, S[\nu(\tau), \omega(t)], \qquad y_1(t) = \rho(\tau)\, G[\nu(\tau), \omega(t)] \\
&\frac{\partial^2 y_0}{\partial \tau^2} = A''S(\nu,\omega) + (2A'\nu' + A\nu'')\frac{\partial S(\nu,\omega)}{\partial \nu} + A\nu'^2 \frac{\partial^2 S(\nu,\omega)}{\partial \nu^2} \\
&\frac{\partial y_1}{\partial \omega} = \rho \frac{\partial G(\nu,\omega)}{\partial \omega}, \qquad \frac{\partial^2 y_1}{\partial \omega \partial \tau} = \rho' \frac{\partial G(\nu,\omega)}{\partial \omega} + \rho\nu' \frac{\partial^2 G(\nu,\omega)}{\partial \nu \partial \omega} \\
&\frac{\partial^2 y_1}{\partial \tau^2} = \rho'' G(\nu,\omega) + (2\rho'\nu' + \rho\nu'')\frac{\partial G(\nu,\omega)}{\partial \nu} + \rho\nu'^2 \frac{\partial^2 G(\nu,\omega)}{\partial \nu^2}
\end{aligned}\right\} \quad (2.13)$$

From the conditions of our theorem and lemma 2 it follows that all the magnitudes entering formulas (2.13) are bounded for $0 \leq t \leq T_0/\varepsilon$. Turning to the expression for $\Delta(t)$ (see (1.3)), we see that it is not difficult to compute a number P such that for $0 \leq t \leq T_0/\varepsilon$

$$|\Delta(t)| \leq P \qquad (2.14)$$

From inequality (2.14) in virtue of equation (1.3) the assertion of our theorem follows.

The above-proven theorem permits us to surmise that the functions

$$y_0(t) = A(\tau)\; S[\nu(\tau),\, \omega(\tau)], \qquad \left(\frac{dy}{dt}\right)_0 = A(\tau)\varphi(\tau)\; \frac{\partial S[\nu(\tau),\, \omega(t)]}{\partial\omega} \tag{2.15}$$

are respectively the principal terms of the asymptotic expansions of the solution of equation (0.1) and of its derivative.

In order to prove this, we set

$$y(t) = y_0(\tau,\omega) + \varepsilon y_1(\tau,\omega) + \varepsilon Y(t), \qquad Y\Big|_{t=0} = 0, \qquad \frac{dY}{dt}\Big|_{t=0} = 0 \tag{2.16}$$

Substituting this expression in (1.2) and using equations (1.4), we obtain

$$\frac{d^2Y}{dt^2} = \Phi(t,Y), \qquad Y\Big|_{t=0} = 0, \qquad \frac{dY}{dt}\Big|_{t=0} = 0 \tag{2.17}$$

where

$$\Phi(t,Y) = -\Big[\varepsilon\Delta(t) + (a - 3by_0^2 - \varepsilon 6by_0y_1 - 3by_1^2\varepsilon^2)Y - (\varepsilon 3by_0 + \varepsilon^2 3by_1)Y^2 - \varepsilon^2 bY^3\Big]$$

From inequality (2.14) we have the result that for $0 \le t \le T_0/\varepsilon$

$$|\Phi(t,0)| \le \varepsilon P \tag{2.18}$$

Let us denote by W a certain positive number. For the conditions of theorem I the functions $y_0(t)$ and $y_1(t)$ are bounded. Hence, if we consider the function $\Phi(t,Y)$ for

$$|Y| < W \tag{2.19}$$

then for $0 \le t \le T_0/\varepsilon$ the function $\Phi(t,y)$ satisfies the Lipshitz condition with respect to Y:

$$|\Phi(t,Y'') - \Phi(t,Y')| \le L|Y'' - Y'| \tag{2.20}$$

We formulate the theorem.

Theorem II: If the conditions of theorem I are satisfied, then for

$$0 \leq \varepsilon \leq \varepsilon_0, \qquad 0 \leq t \leq T(\varepsilon), \qquad \left(T(\varepsilon) = \frac{1}{\sqrt{L}} \ln\left[1 + \frac{LW}{P}\frac{1}{\varepsilon}\right]\right)$$

the following inequalities hold:

$$|y(t) - A(\tau)S(\nu,\omega)| \leq H\varepsilon, \qquad \left|\frac{dy}{dt} - A(\tau)\varphi(\tau)\frac{\partial S(\nu,\omega)}{\partial\omega}\right| \leq H_t\varepsilon$$

To prove this we rewrite equation (2.17) in the form

$$Y = \int_0^t \int_0^{t_1} \Phi(\eta,Y)d\eta\, dt_1 \tag{2.21}$$

and apply to it the method of successive approximations:

$$Y^{(0)} = \int_0^t \int_0^{t_1} \Phi(\eta,0)d\eta\, dt_1, \ . \ . \ .,$$

$$Y^{(n+1)} = \int_0^t \int_0^{t_1} \Phi(\eta,Y^{(n)})d\eta\, dt_1$$

Estimating the functions $Y^{(0)}$, $[Y^{(n+1)} - Y^{(n)}]$ with the aid of inequalities (2.18) and (2.20), we obtain

$$|Y^{(0)}| \leq \varepsilon P\frac{t^2}{2!}, \qquad |Y^{(n+1)} - Y^{(n)}| \leq \varepsilon PL^{n+1}\frac{l^{2n+4}}{(2n+4)!} \tag{2.22}$$

These estimates are valid for the condition that $|Y^{(n+1)}(t)|$ does not exceed W. Below we determine a number $T(\varepsilon) \leq T_0/\varepsilon$ such that for $0 \leq t \leq T(\varepsilon)$ and $0 \leq \varepsilon \leq \varepsilon_0$ this condition will always be satisfied. The function $Y(t)$ can be represented in the form

$$Y(t) = Y^{(0)} + \sum_{n=0}^{\infty}[Y^{(n+1)} - Y^{(0)}] \tag{2.23}$$

Applying inequalities (2.22) for $0 \leq t \leq T(\varepsilon)$ and $0 \leq \varepsilon \leq \varepsilon_0$, we obtain

$$|Y(t)| \leq \varepsilon\frac{P}{L}(e^{\sqrt{L}\,t} - 1) \tag{2.24}$$

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We now determine the number $T(\varepsilon)$. In virtue of the equation

$$Y^{(n+1)} = Y^{(0)} + \sum_{n=0}^{n+1} [Y^{(n+1)} - Y^{(n)}]$$

we have $|Y^{(n+1)}(t)| \leq \varepsilon(P/L[\exp(\sqrt{L}\, t) - 1]$. Hence if we determine the number $T(\varepsilon)$ such that $\varepsilon(P/L)[\exp(\sqrt{L}\, T(\varepsilon)) - 1] = W$, then $|Y^{(n+1)}(t)|$ for $0 \leq t \leq T(\varepsilon)$ and $0 \leq \varepsilon \leq \varepsilon_0$ will not exceed W. We thus have

$$T(\varepsilon) = \frac{1}{\sqrt{L}} \ln \left(1 + \frac{LW}{P}\frac{1}{\varepsilon}\right)$$

We choose the number ε_0 such that

$$\frac{1}{\sqrt{L}} \ln \left(1 + \frac{LW}{P}\frac{1}{\varepsilon_0}\right) \leq \frac{T_0}{\varepsilon_0}$$

Then, evidently $T(\varepsilon) \leq T_0/\varepsilon$ for $0 \leq \varepsilon \leq \varepsilon_0$.

From (2.21) we have

$$\left|\frac{dY}{dt}\right| \leq \int_0^t |\Phi(t,0)|dt + \int_0^t |\Phi(t,Y) - \Phi(t,0)|dt \qquad (2.25)$$

Using inequalities (2.18), (2.20), and (2.24) for $0 \leq t \leq T(\varepsilon)$ and $0 \leq \varepsilon \leq \varepsilon_0$, we obtain

$$\left|\frac{dY}{dt}\right| \leq \varepsilon tP + L\int_0^t |Y|dt \leq \varepsilon\frac{P}{\sqrt{L}}(e^{-\sqrt{L}\,t} - 1) \leq \varepsilon\frac{P}{\sqrt{L}}(e^{\sqrt{L}\,T(\varepsilon)} - 1) = \sqrt{L}W$$

(2.26)

From (2.16) we have

$$\left.\begin{aligned} &Y(t) - A(\tau)\, S(\nu,\omega) = \varepsilon[\rho G(\nu,\omega) + Y(t)] \\ &\frac{dY}{dt} - A(\tau)\varphi(\tau)\frac{\partial S(\nu,\omega)}{\partial\omega} = \varepsilon\left[A'S(\nu,\omega) + A\nu'\frac{\partial S(\nu,\omega)}{\partial\nu} + \rho\varphi\frac{\partial G(\nu,\omega)}{\partial\omega} + \right. \\ &\qquad \left. \frac{dY}{dt} + \varepsilon\rho\nu'\frac{\partial G(\nu,\omega)}{\partial\nu} + \varepsilon\rho' G(\nu,\omega)\right] \end{aligned}\right\}$$

(2.27)

In virtue of the conditions of the proven theorem, lemma 2, and inequalities (2.24) and (2.26), the functions in brackets on the right sides of equations (2.27) are bounded for $0 \leq t \leq T(\varepsilon)$ and $0 \leq \varepsilon \leq \varepsilon_0$. Setting

$$H = \max\,[\rho G(\nu,\omega) + Y(t)]$$

$$H_t = \max\left[A'S(\nu,\omega) + A\nu' \frac{\partial S(\nu,\omega)}{\partial \nu} + \rho\varphi \frac{\partial G(\nu,\omega)}{\partial \omega} + \frac{dY}{dt} + \varepsilon\rho\nu' \frac{\partial G(\nu,\omega)}{\partial \nu} + \varepsilon\rho' G(\nu,\omega)\right]$$

from formulas (2.27) we obtain the inequalities

$$|y(t) - A(\tau)S(\nu,\omega)| \leq H\varepsilon, \qquad \left|\frac{dy}{dt} - A(\tau)\varphi(\tau)\frac{\partial S(\nu,\omega)}{\partial \omega}\right| \leq H_t\varepsilon$$

valid for $0 \leq t \leq T(\varepsilon)$ and $0 \leq \varepsilon \leq \varepsilon_0$. Thus the theorem has been proved.

<u>3. Investigation of the formulas for the functions $\nu(\tau)$, $A(\tau)$, and $\varphi(\tau)$.</u> - Let $\psi(\tau) = K(\nu(\tau))\varphi(\tau)$; then the system (1.8) and (1.28) assumes the form

$$\psi^2(\tau)(1 + \nu(\tau)) = a(\tau), \qquad \psi^2(\tau)2\nu(\tau) = b(\tau)A^2(\tau), \qquad \psi(\tau)A^2(\tau)L(\nu(\tau)) = B \tag{3.1}$$

where B is a constant. The function $L(\nu)$(see (1.27)) can be expressed in terms of elliptic integrals (ref. 5)

$$L(\nu) = \frac{(1 + \nu)E(\nu) - (1 - \nu)K(\nu)}{3\nu}$$

From the first, third, and then second equations of (3.1) successively, we have

$$\psi^2(\tau) = \frac{a(\tau)}{1 + \nu(\tau)}, \qquad A^2(\tau) = \sqrt{\frac{1 + \nu(\tau)}{a(\tau)}}\,\frac{B}{L(\nu(\tau))},$$

$$\frac{4\nu^2(\tau)L^2(\nu(\tau))}{(1 + \nu(\tau))^3} = \frac{B^2b^2(\tau)}{a^3(\tau)} \tag{3.2}$$

We dwell on the case where the equation (0.1) is linear and $a(\tau) > 0$. Assuming in formulas (3.2) $b(\tau) \equiv 0$, we obtain

$$\nu(\tau) \equiv 0, \qquad \psi_l^2(\tau) = a(\tau), \qquad A_l^2(\tau) = \frac{B}{L(0)}\,\frac{1}{\sqrt{a(\tau)}} \tag{3.3}$$

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Formulas (3.3) agree with the formulas obtained in reference 4. Using (3.3), the first two formulas of (3.2) can be written in the form

$$\psi^2(\tau) = \psi_l^2(\tau)\,\frac{1}{1+\nu(\tau)}, \qquad A^2(\tau) = A_l^2(\tau)\,\frac{L(0)\sqrt{1+\nu(\tau)}}{L(\nu(\tau))} \tag{3.4}$$

where the subscript l denotes the magnitudes referred to the linear equation.

These formulas make it easier to see the part played by the nonlinear term in equation (0.1).

Let us consider the case where $a(\tau) > 0$, $b(\tau) \geq 0$. The system described by equation (0.1) may now lose stability. This evidently can occur at the moment when $a(\tau) = b(\tau)A^2(\tau)$.

From the first two equations of the system (3.1) it follows that in this case $\nu(\tau) = 1$. Making use of this fact we obtain from the last of equations (3.2) the condition of stability

$$\frac{B^2b^2(\tau)}{a^3(\tau)} < \frac{4L^2(1)}{2^3} = \frac{2}{9}, \qquad \left(L(1) = \int_0^1 (1-\zeta^2)d\zeta = \frac{2}{3}\right) \tag{3.5}$$

Since the constant B is determined by the initial conditions $B = B(y_{t=0}, (dy/dt)_{t=0})$, we can, with the aid of condition (3.5) for the given time segment $[0, t_1]$, determine in the phase plane a region σ such that for $t = 0$ the point represented is within σ; then for $0 \leq t \leq t_1$ the system does not lose its stability. The region σ evidently is determined by the inequality

$$B^2\left(y\Big|_{t=0}, \quad \frac{dy}{dt}\Big|_{t=0}\right) < \frac{2}{9}\min\left[\frac{a^3(\tau)}{b^2(\tau)}\right] \tag{3.6}$$

In order to compare the stability conditions obtained by us with the stability condition obtained from the quasi-static consideration of equation (0.1), let us analyze the case

$$y\Big|_{t=0} = \alpha, \qquad \frac{dy}{dt}\Big|_{t=0} = 0, \qquad b(0) = 0 \tag{3.7}$$

where for simplicity we assume that

$$\min\left[\frac{a^3(\tau)}{b^2(\tau)}\right] = \frac{[\min\, a(\tau)]^3}{[\max\, b(\tau)]^2} \tag{3.8}$$

For the analysis we make use of formulas (2.15). Substituting in them

$$\omega(t) = \int_0^t \frac{\psi(\tau)}{K(\nu(\tau))}\, dt + \omega_0 \tag{3.9}$$

we rewrite them in the form

$$\left.\begin{aligned} y_0(t) &= A(\tau)\,\mathrm{sn}\, K(\nu)\left[\int_0^t \frac{\psi(\tau)}{K(\nu)}\, dt + \omega_0\right] \\ \left(\frac{dy}{dt}\right)_0 &= A(\tau)\psi(\tau)\,\mathrm{cn}\, K(\nu)\left[\int_0^t \frac{\psi(\tau)}{K(\nu)}\, dt + \omega_0\right]\mathrm{dn}\, K(\nu)\left[\int_0^t \frac{\psi(\tau)}{K(\nu)}\, dt + \omega_0\right]\end{aligned}\right\} \tag{3.10}$$

Setting in them $t = 0$, we obtain $\omega_0 = 1$, $A(0) = \alpha$.

Since $b(0) = 0$, the second and third formulas of (3.2) for $t = 0$ give

$$\nu(0) = 0, \qquad B(\alpha) = A^2(0)\sqrt{a(0)}\, L(0) = \frac{1}{4}\pi\sqrt{a(0)}\,\alpha^2$$

Substituting $B(\alpha)$ in condition (3.6) and using (3.8), we obtain

$$\alpha^2 < \frac{4}{\pi}\frac{\sqrt{2}}{3}\sqrt{\frac{\min a(\tau)}{a(0)}}\,\frac{\min a(\tau)}{\max b(\tau)} \tag{3.11}$$

The stability condition obtained from the quasi-static consideration of equation (0.1) has the following form:

$$\alpha^2 < \frac{\min a(\tau)}{\max b(\tau)}$$

From the fact that the multiplier on the right side of the inequality (3.11)

$$\frac{4}{\pi}\frac{\sqrt{2}}{3}\sqrt{\frac{\min a(\tau)}{a(0)}} < 1$$

it follows that a system that is stable from the quasi-static point of view may actually turn out to be unstable.

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